

FRACALMO PRE-PRINT: www.fracalmo.org

Mathematical Methods in Economics and Finance: Vol. 1 (2006) 37-55

UNIVERSITÀ CA' FOSCARI DI VENEZIA - DIPARTIMENTO DI MATEMATICA APPLICATA

ISSN: 1971-6419 (PRINT EDITION). ISSN: 1971-3878 (ELECTRONIC EDITION)

URL: <http://www.dma.unive.it/mmef/>

The origin of infinitely divisible distributions: from de Finetti's problem to Lévy-Khintchine formula

Francesco MAINARDI ⁽¹⁾ and Sergei ROGOSIN ⁽²⁾

⁽¹⁾ Department of Physics, University of Bologna, and INFN,
Via Irnerio 46, I-40126 Bologna, Italy

Corresponding Author. E-mail: francesco.mainardi@unibo.it

⁽²⁾ Department of Mathematics and Mechanics, Belarusian State University,
Nezavisimosti Ave 4, BY-220030, Minsk, Belarus
E-mail: rogosin@bsu.by

Abstract

The article provides an historical survey of the early contributions on infinitely divisible distributions starting from the pioneering works of de Finetti in 1929 up to the canonical forms developed in the thirties by Kolmogorov, Lévy and Khintchine. Particular attention is paid to single out the personal contributions of the above authors that were published in Italian, French or Russian during the period 1929-1938. In Appendix we report the translation from the Russian into English of a fundamental paper by Khintchine published in Moscow in 1937.

Keywords. Characteristic function, infinitely divisible distributions, stochastic processes with independent increments, de Finetti, Kolmogorov, Lévy, Khintchine, Gnedenko.

M.S.C. classification. 60E07, 60E10, 60G51, 01A70.

J.E.L. classification. C10, C16.

1 Introduction

The purpose of this paper is to illustrate how the concept of an infinitely divisible distribution has been developed up to obtain the canonical form of its characteristic function.

Usually historical aspects on this development are known thanks to some notes available in the classical textbooks by Lévy [75] (published in French in 1937 and 1954), by Gnedenko-Kolmogorov [45] (published in Russian in 1949 and translated into English in 1954) and by Feller [38] (published in English in 1966 and 1971). Similar historical notes can be extracted from the recent treatises by Sato [105] and by Steutel and van Harn [112].

In our opinion, however, a better historical analysis can be accomplished if one examines the original works of the pioneers, namely Bruno de Finetti (1906-1985) [26, 27, 28, 29, 31], Andrei Nikolaevich Kolmogorov (1903-1987) [68, 69], who published in Italian in *Rendiconti della R. Accademia Nazionale dei Lincei*, Paul Lévy (1886-1971) [73, 74], who published in French in *Annali della R. Scuola Normale di Pisa*, and finally Alexander Yakovlevich Khintchine¹ (1894-1959) [62], who published in Russian in the *Bulletin of the Moscow State University*. Noteworthy is the 1938 book by Khintchine himself [65], in Russian, on *Limit Distributions for Sums of Independent Random Variables*. For the reader interested in the biographical notes and bibliography of the mentioned scientists we refer: for de Finetti to [20, 23, 24, 25], for Kolmogorov to [108, 109], for Lévy to [76, 77] and for Khintchine to [44].

In spite of the fact that de Finetti was the pioneer of the infinitely divisible distributions in view of his 1929-1931 papers, as is well recognized in the literature, the attribute *infinitely divisible*, as noted by Khintchine in his 1938 book [65], first appeared in the Moscow mathematical school, precisely in the 1936 unpublished thesis by G.M. Bawly (1908-1941)². According to Khintchine [65] the name of *infinitely divisible distributions* (in a printed version) is found in the 1936 article by G.M. Bawly [5], that was recommended for publication in the very important starting volume of the new series of *Matematicheski Sbornik*. However, we note that this term was not “stably”

¹There is also the transliteration Khinchin.

²Gregory Minkelevich Bawly graduated at the Moscow State University in 1930, defended his PhD thesis under guidance of A.N. Kolmogorov in 1936. His scientific advisor had greatly esteemed his results on the limit distributions for sums of independent random variables and cited him in his book with Gnedenko [45]. G.M. Bawly lost his life in Moscow in November 1941 at a bombing attack.

applied in the article. Two alternative (and equivalent) terms were used, namely infinitely = unbeschränkt (German) and unboundedly = unbegrenzt (German), see [5, p. 918].

The first formal definition of an infinitely divisible distribution was given by Khintchine himself [63]. It reads: *a distribution of a random variable which for any positive integer n can be represented as a sum of n identically distributed independent random variables is called an infinitely divisible distribution.*

We note that infinitely divisible distributions (already under this name) were formerly studied systematically in the 1937 book by Lévy [75], and soon later in the 1938 book by Khintchine [65]. We also note that Lévy himself, in his late biographical 1970 booklet [76, p. 103], attributes to Khintchine the name *indéfiniment divisible*. The canonical form of infinitely divisible distributions is known in the literature as *Lévy-Khintchine formula*, surely because it was so named by Gnedenko and Kolmogorov [45] in their classical treatise on *Limit Distributions for Sums of Independent Random Variables*³ that has appeared in Russian in 1949 and in English in 1954.

The plan of the present paper is as follows. In Section 2 we provide a survey of the known results on infinitely divisible distributions. Then we pass to present the tale on the origin of these results by recalling, in a historical perspective, the early publications of our four actors: de Finetti, Kolmogorov, Lévy and Khintchine. Section 3 is devoted to de Finetti and Kolmogorov, namely to the so-called de Finetti's problem (as it was referred to by Kolmogorov). Section 4 is devoted to Lévy and Khintchine, namely to the origin of the so-called Lévy-Khintchine formula.

To our knowledge the original contributions by Khintchine have never been translated into English, so we find it convenient to report in Appendix the English translation of his 1937 paper, that has led to the Lévy-Khintchine formula. We plan to publish the English translation of the 1938 book by Khintchine on *Limit Distributions for Sums of Independent Random Variables* along with a few related articles of him (originally in Italian, German and Russian).

Concerning our bibliography, the main text and the footnotes give references to some classical publications. However, we take this occasion to edit

³We note that the Russian titles of both books by Khintchine and Gnedenko & Kolmogorov are identical, although in the reference list (in Russian and in English) of the book by Gnedenko & Kolmogorov the title of Khintchine's previous book is in some way different (*Limit Theorems for Sums of Independent Random Variables*).

a more extended bibliography on infinite divisible distributions and related topics, that, even if non-exhaustive, could be of some interest.

2 A survey on infinitely divisible distributions

Hereafter we recall the classical results on infinitely divisible distributions just to introduce our notations. We presume that the reader has a good knowledge in the Probability Theory. In the below formulations we essentially follow the treatments by Feller [38] and by Lukacs [80]; in the references, however, we have cited several treatises containing excellent chapters on infinite divisible distributions.

A probability distribution F is infinitely divisible iff for each $n \in \mathbb{N}$ it can be represented as the distribution of the sum

$$S_n = X_{1,n} + X_{2,n} + \dots + X_{n,n} \quad (2.1)$$

of n independent random variables with a common distribution F_n . It is common to locate the random variables in an infinite triangular array

$$\begin{array}{ccccccc} X_{1,1} & & & & & & \\ X_{2,1}, & X_{2,2} & & & & & \\ X_{3,1}, & X_{3,2}, & X_{3,3} & & & & \\ & \dots & & & & & \\ X_{n,1}, & X_{n,2}, & X_{n,3}, & \dots, & X_{n,n} & & \\ & \dots & & & \dots & & \end{array} \quad (2.2)$$

whose rows contain independent identically distributed (*iid*) random variables.

This definition is valid in any number of dimensions, but for the present we shall limit our attention to one-dimensional distributions. It should be noted that infinite divisibility is a property of the *type*, that is, together with F all distributions differing from F only by location parameters are infinitely divisible. *Stable* distributions (henceforth the Gaussian and the Cauchy distributions) are infinitely divisible and distinguished by the fact that F_n differs from F only by location parameters.

On account of the convolution property of the distribution functions of independent random variables, the distribution function F turns out to be

the n -fold convolution of some distribution function F_n ; then, the notion of infinite divisibility can be introduced by means of the *characteristic function*:

$$\varphi(t) := \mathbb{E}\{e^{itX}\} := \int_{-\infty}^{+\infty} e^{itx} dF(x). \quad (2.3)$$

In fact, for an infinitely divisible distribution its characteristic function $\varphi(t)$ turns out to be, for every positive integer n , the n -th power of some characteristic function. This means that there exists, for every positive integer n , a characteristic function $\varphi_n(t)$ such that

$$\varphi(t) = [\varphi_n(t)]^n. \quad (2.4)$$

The function $\varphi_n(t)$ is uniquely determined by $\varphi(t)$, $\varphi_n(t) = [\varphi(t)]^{1/n}$, provided that one selects the principal branch for the n -th root.

Since Eqs. (2.2) and (2.4) are equivalent, alternatively one could speak about infinitely divisible distributions or infinitely divisible characteristic functions. Elementary properties of infinitely divisible characteristic functions are listed by Lukacs [80]. The concept of infinite divisibility is very important in probability theory, particularly in the study of limit theorems.

Here we stress the fact that infinitely divisible distributions are intimately connected with *stochastic processes with independent increments*. By this we mean a family of random variables $X(\lambda)$ depending on the continuous parameter λ and such that the increments $X(\lambda_{k+1}) - X(\lambda_k)$ are mutually independent for any finite set $\{\lambda_1 < \lambda_2 < \dots < \lambda_n\}$. More precisely the processes are assumed to be *homogeneous*, that is with *stationary increments*. Then the distribution of $Y(\lambda) := X(\lambda_0 + \lambda) - X(\lambda_0)$ depends only on the length λ of the interval but not on λ_0 . Let us make a partition the interval $[\lambda_0, \lambda_0 + \lambda]$ by $n + 1$ equidistant points $\lambda_0 < \lambda_1 < \dots < \lambda_n = \lambda_0 + \lambda$ and put $X_{k,n} = X(\lambda_k) - X(\lambda_{k-1})$. Then the variable $Y(\lambda)$ of a process with stationary independent increments is the sum of n independent variables $X_{k,n}$ with a common distribution and hence $Y(\lambda)$ *has an infinitely divisible distribution*. We can summarise all above by simply writing the characteristic function of $Y(\lambda)$ for any $\lambda > 0$ as

$$\varphi(t, \lambda) := \mathbb{E}\{e^{itX(\lambda)}\} = \{\varphi(t, 1)\}^\lambda. \quad (2.5)$$

We note that we have adopted the notation commonly used in the early contributions: the letter t denotes the Fourier parameter of the characteristic function whereas the continuous parameter (essentially the time) of a

stochastic process has been denoted by the letter λ . Only later, when the theory of stochastic processes became well developed, the authors had denoted the Fourier parameter by a different letter like u or κ reserving, as natural, the letter t to the time entering the stochastic processes. The reader should be aware of the old notation in order to avoid possible confusion.

Let us close this section by recalling (essentially based on the book by Lukacs) the main theorems concerning the structure of infinitely divisible distributions, that are relevant to our historical survey.

First de Finetti's Theorem: A characteristic function is infinitely divisible iff it has the form

$$\varphi(t) = \lim_{m \rightarrow \infty} \exp\{p_m[\psi_m(t) - 1]\}, \quad (2.6)$$

where the p_m are real positive numbers while $\psi_m(t)$ are characteristic functions.

Second de Finetti's Theorem: The limit of a sequence of finite products of Poisson-type characteristic functions is infinitely divisible. The converse is also true. This means that the class of infinitely divisible laws coincides with the class of distribution limits of finite convolutions of distributions of Poisson-type⁴.

The Kolmogorov canonical representation: The function $\varphi(t)$ is the characteristic function of an infinitely divisible distribution with finite second moment iff it can be written in the form

$$\log \varphi(t) = i\gamma t + \int_{-\infty}^{+\infty} (e^{itu} - 1 - itu) \frac{dK(u)}{u^2}, \quad (2.7)$$

where γ is a real constant, and $K(x)$ is a non-decreasing and bounded function such that $K(-\infty) = 0$. The integrand is defined for $u = 0$ to be equal to $-(t^2/2)$.

⁴Let us recall that for the characteristic function of the Poisson distribution we have according to (2.4)

$$\varphi(t) = \exp[\lambda(e^{it} - 1)], \quad \text{so that} \quad \varphi_n(t) = \exp\left[\frac{\lambda}{n}(e^{it} - 1)\right].$$

The theorem can be used to show that a given characteristic function is infinitely divisible. For an example we refer the reader to [80, p. 113].

The Lévy canonical representation: The function $\varphi(t)$ is the characteristic function of an infinitely divisible distribution iff it can be written in the form

$$\log \varphi(t) = i\gamma t - \frac{\sigma^2}{2} t^2 + \int_{-\infty}^{-0} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) dM(u) + \int_{+0}^{+\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) dN(u), \quad (2.8)$$

where γ is a real constant, σ^2 is a real and non-negative constant, and the functions $M(u)$, $N(u)$ satisfy the following conditions:

- (i) $M(u)$ and $N(u)$ are non-decreasing in $(-\infty, 0)$ and $(0, +\infty)$, respectively.
- (ii) $M(-\infty) = N(+\infty) = 0$.
- (iii) The integrals $\int_{-\epsilon}^0 u^2 dM(u)$ $\int_0^{+\epsilon} u^2 dN(u)$ are finite for every $\epsilon > 0$.

The Lévy-Khintchine canonical representation: The function $\varphi(t)$ is the characteristic function of an infinitely divisible distribution iff it can be written in the form

$$\log \varphi(t) = i\gamma t + \int_{-\infty}^{+\infty} \left[e^{itu} - 1 - \frac{itu}{1+u^2} \right] \frac{1+u^2}{u^2} dG(u), \quad (2.9)$$

where γ is a real constant, and $G(u)$ is a non-decreasing and bounded function such that $G(-\infty) = 0$. The integrand is defined for $u = 0$ to be equal to $-(t^2/2)$.

We point out that there is a tight connection between the Lévy-Khintchine canonical representation and the general Central Limit Theorem. For a clear description of a modern view on this connection we refer, *e.g.*, to [49].

3 The work of de Finetti and Kolmogorov

Bruno de Finetti is recognized to be the most prominent scientist of the Italian school of Probability and Statistics, that started at the beginning of the last century with Guido Castelnuovo (1865-1952) and Francesco Paolo Cantelli (1875-1966). His personality and his interest in probability came out already with his attendance at the 1928 International Congress of Mathematicians⁵ held in Bologna (Italy) from 3 to 10 September 1928. The young

⁵The Chairman of the Congress was Salvatore Pincherle (1853-1936), Professor of Mathematics at the University of Bologna from 1880 up to 1928, the year of the Congress. He was the first President of the Unione Matematica Italiana (UMI) from 1922 up to 1936, at his death.

de Finetti presented a note on the role of the characteristic function in random phenomena [33]⁶, that was published only in 1932 in the Proceedings of the Congress⁷.

At the Bologna Congress de Finetti had the occasion to meet Lévy and Khintchine (who were included in the French and Russian delegations, respectively) but we are not informed about their interaction. We note, however, that Khintchine did not present any communication whereas Lévy presented a note outside the field of probability, precisely on fractional differentiation [72]; furthermore Kolmogorov did not attend the Congress. Surely Lévy, Kolmogorov and Khintchine held in high consideration the Italian school of Probability since in the thirties they submitted some relevant papers to Italian journals (written in Italian for the Russians and in French for Lévy), see *e.g.* [59, 60, 61, 68, 69, 73, 74].

Just after the Bologna Congress de Finetti started a research regarding functions with random increments, see [26, 27, 28, 29, 31] based on the theory of infinitely divisible characteristic functions, even if he did not use such term. His results can be summarized in a number of relevant theorems (partly stated in the previous Section). As it was already mentioned they are highly connected with the stochastic processes with stationary independent increments. In this respect we refer the reader to the Section 2.2 of the excellent paper by Cifarelli and Regazzini on de Finetti's contributions in Probability and Statistics [20].

The papers by de Finetti, published in the period 1929-1931 (in Italian) in the Proceedings of the Royal Academy of Lincei (*Rendiconti della Reale Accademia Nazionale dei Lincei*) [26, 27, 28, 29, 31], attracted the attention of Kolmogorov who was interested to solve the so-called **de Finetti's problem**, that is the problem of finding the general formula for the characteristic function of the infinitely divisible distributions. This problem was indeed attacked by Kolmogorov in 1932 in two notes published in Italian in the

⁶We have to mention that this work by de Finetti was the first significant contribution to the subject known now as the theory of exchangeable sequences (of events). A more exhaustive account appeared in 1931 in [30].

⁷The Proceedings were published by Zanichelli, Bologna, with all the details of the scientific and social programs, in 6 volumes, that appeared from 1929 to 1932. The papers, published in one of the following languages: Italian, French, German and English, were classified in 7 sessions according to their topic. The papers presented by Cantelli [17], de Finetti, Romanovsky and Slutsky (Session IV, devoted to Actuarial Sciences, Probability and Statistics) were included within the last volume, published in 1932.

same journal as de Finetti (Rendiconti della Reale Accademia Nazionale dei Lincei), where he gave an exhaustive answer to **de Finetti's problem** for the case of variables with *finite second moment*, see [68, 69]. These two notes are available in English in a unique paper (No 13) in the *Selected Works of A.N. Kolmogorov* with a comment of V.M. Zolotarev, see [70]: the final result of Kolmogorov is reported in Section 2 as Eq. (2.7), known as **the Kolmogorov canonical representation** of the infinitely divisible characteristic functions.

4 The work of Lévy and Khintchine

The general case of de Finetti's problem, including also the case of *infinite variance*, was investigated in 1934-35 by Lévy [73, 74] who published two papers in French in the Italian Journal: Annali della Reale Scuola Normale di Pisa. At that time Lévy was quite interested in the so-called *stable distributions* that are known to exhibit infinite variance, except for the particular case of the Gaussian.

The approach by Lévy, well described in his classical 1937 book [75], is quite independent from that of Kolmogorov, as can be understood from footnotes in his 1934 paper [73], that we report partly below in original. From the footnote⁽¹⁾ we learn that the results contained in his paper were presented in three communications of the Academy of Sciences (Comptes Rendus) of 26 February, 26 March and 7 May 1934. Then, in the footnote⁽⁶⁾, p. 339, the Author writes:

[Ajouté à la correction des épreuves] Le résumé de ma note du 26 février, rédigé par M. Kolmogorov, a attiré mon attention sur deux Notes de M. B. de Finetti (see [26, 29]) et deux autres de M. Kolmogorov lui-même (see [68, 69]), publiées dans les Atti Accademia Naz. Lincei (VI ser). Ces dernières notamment contiennent la solution du problème traité dans le présent travail, dans le cas où le processus est homogène et où la valeur probable $\mathbb{E}\{x^2\}$ est finie. Le résultat fondamental du présent Mémoire apparaît donc comme une extension d'un résultat de M. Kolmogorov.

This means that P. Lévy was not aware about the results on homogeneous processes with independent increments obtained by B. de Finetti and by A. N. Kolmogorov. The final result of Lévy is reported in Section 2 as Eq. (2.8), known as **the Lévy canonical representation** of the infinitely divisible characteristic functions.

In a paper of 1937 Khintchine [62] showed that Lévy's result can be obtained also by an extension of Kolmogorov's method: his final result, reported in Section 2 as Eq. (2.9), is known as **the Lévy-Khintchine canonical representation** of the infinitely divisible characteristic functions. The translation from the Russian of this fundamental paper can be found in Appendix. The theory of infinitely distributions was then presented in German in the article [64] and in Russian in his 1938 book on *Limit Distributions for Sums of Independent Random Variables* [65].

Unfortunately, many contributions by Khintchine (being in Russian) remained almost unknown in the West up to the English translation of the treatise by Gnedenko and Kolmogorov [45] in 1954.

The obituary of Khintchine [44], that Gnedenko (his former pupil) presented at the 1960 Berkeley Symposium on Mathematical Statistics and Probability, provides a general description of the works of Khintchine along with a complete bibliography. From that we learn that the 1938 book by Khintchine was preceded by a special course of lectures in Moscow University that attracted the interest of A.A. Bobrov, D.A. Raikov and B.V. Gnedenko himself.

Acknowledgements

The authors are grateful to R. Gorenflo and the anonymous referees for useful comments. We thank also O. Celebi for the help with the paper by G.M. Bawly published in Turkey.

Appendix: Khintchine's 1937 article

A. Ya. Khintchine⁸ : A new derivation of a formula by P. Lévy, *Bulletin of the Moscow State University* **1** (1937) 1-5.

A collection of all the so-called infinitely divisible distributions was discovered for the first time by P. Lévy⁹. He has derived a remarkable formula for the logarithm of the characteristic function of such a distribution. Because of

⁸We have to remark that the footnotes in this Appendix are translation of the original ones by Khintchine.

⁹Ann. R. Scuola Norm. Pisa (Ser. II), **3**, pp. 337-366 (1934).

the importance of this formula I shall give here a new completely analytic and very simple proof of it¹⁰.

Let $\varphi(x)$ be an infinitely divisible distribution and let $\varphi(t)$ be the corresponding characteristic function. It is known that for each $h \geq 0$ the function $\varphi(t)^h$ is a characteristic function as well. We denote by $\varphi_h(x)$ the corresponding distribution. Thus

$$\log \varphi(t) = \lim_{h \rightarrow 0} \frac{\varphi(t)^h - 1}{h} = \lim_{h \rightarrow 0} I_h(t),$$

where

$$I_h(t) = \frac{1}{h} \int_{-\infty}^{+\infty} (e^{itu} - 1) d\varphi_h(u).$$

Put for each $h > 0$

$$G_h(u) = \int_0^u \frac{v^2}{v^2 + 1} \frac{d\varphi_h(v)}{h}. \quad (A.1)$$

Clearly the function $G_h(u)$ is nondecreasing and bounded. Furthermore,

$$I_h(t) = \int_{-\infty}^{+\infty} (e^{itu} - 1) \frac{u^2 + 1}{u^2} dG_h(u).$$

Taking the real part of this formula we have

$$-\operatorname{Re} I_h(t) = \int_{-\infty}^{+\infty} (1 - \cos tu) \frac{u^2 + 1}{u^2} dG_h(u). \quad (A.2)$$

Let

$$A_h := \int_{|u| \leq 1} dG_h(u), \quad B_h := \int_{|u| > 1} dG_h(u), \quad C_h := A_h + B_h = \int_{-\infty}^{+\infty} dG_h(u).$$

Relation (A.2) gives us for $t = 1$

$$-\operatorname{Re} I_h(1) \geq \int_{|u| \leq 1} (1 - \cos u) \frac{u^2 + 1}{u^2} dG_h(u) \geq cA_h, \quad (A.3)$$

¹⁰The method of this proof can be considered as an extension of the idea by A. N. Kolmogorov. The latter formed the base of the proof of an analogous formula in the important case of finite variance (see *Rendiconti dei Lincei*, **15**, pp. 805-808 and 866-869 (1932).

where c is a strictly positive constant. In the same way for each t we have

$$-\operatorname{Re} I_h(t) \geq \int_{|u| \geq 1} (1 - \cos tu) dG_h(u).$$

Hence

$$-\int_0^2 \operatorname{Re} I_h(t) dt \geq 2B_h - \int_{|u| \geq 1} \frac{\sin 2u}{u} dG_h(u) \geq B_h. \quad (\text{A.4})$$

It follows from Eqs. (A.3)-(A.4) that

$$C_h = A_h + B_h = -\frac{\operatorname{Re} I_h(1)}{c} - \int_0^2 \operatorname{Re} I_h(t) dt.$$

Since the function $I_h(t)$ uniformly converges on $0 \leq t \leq 2$ as $h \rightarrow 0$ to a finite limit, then C_h is bounded as $h \rightarrow 0$. Since $G_h(0) = 0$, the functions $G_h(u)$ remain uniformly bounded for $h \rightarrow 0$. Therefore, there exists a sequence of positive numbers h_n ($n = 1, 2, \dots$) such that $h_n \rightarrow 0$ as $n \rightarrow \infty$, and the sequence of functions $G_{h_n}(u)$ converges to a (bounded nondecreasing) function $G(u)$ as $n \rightarrow \infty$. With

$$\gamma_n = \int_{-\infty}^{+\infty} \frac{dG_{h_n}(u)}{u}$$

(where the integral has a sense due to (A.1)), we have

$$\log \varphi(t) = \lim_{n \rightarrow \infty} \left\{ it\gamma_n + \int_{-\infty}^{+\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) \frac{u^2+1}{u^2} dG_{h_n}(u) \right\}.$$

Since the integrand of the above integral is bounded and continuous, this integral tends as $n \rightarrow \infty$ to

$$\int_{-\infty}^{+\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) \frac{u^2+1}{u^2} dG(u).$$

Hence the sequence γ_n should converge to a certain positive constant γ . Therefore,

$$\log \varphi(t) = it\gamma + \int_{-\infty}^{+\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) \frac{u^2+1}{u^2} dG(u). \quad (\text{A.5})$$

This is the P. Lévy formula up to certain unessential details concerning the way of its presentation.

To prove the uniqueness of the last representation it is easier to get the inversion formula. Let

$$\Delta(t) = \int_{t-1}^{t+1} \log \varphi(\alpha) d\alpha - 2 \log [\varphi(t)].$$

Then Eq. (A.5) gives immediately

$$\Delta(t) = -2 \int_{-\infty}^{+\infty} e^{itu} \left(1 - \frac{\sin u}{u}\right) dG(u) = \int_{-\infty}^{+\infty} e^{itu} dK(u),$$

where

$$K(u) = -2 \int_0^u \left(1 - \frac{\sin v}{v}\right) dG(v).$$

Then, the well-known P. Lévy inversion formula¹¹ yields

$$K(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1 - e^{-itu}}{it} \Delta(t) dt.$$

It follows that $K(u)$ (and hence $G(u)$) is completely determined by the function $\varphi(t)$. Then, we can easily conclude that

$$G_h(u) \rightarrow G(u) \quad \text{as } h \rightarrow 0.$$

If this were not true, then there it would exist a sequence of functions $G_h(u)$ converging to another function (different from $G(u)$). This would give another representation of the type (A.5) for the function $\log \varphi(t)$.

Vice-versa, now we can show that if the logarithm of a function $\varphi(t)$ is represented in the form (A.5) for a certain nondecreasing bounded function $G(u)$, then $\varphi(t)$ is a characteristic function of an infinitely divisible distribution. Let ε be an arbitrary positive number. Put

$$\Delta_\varepsilon = G(\varepsilon) - G(-\varepsilon),$$

$$G_\varepsilon(u) = \begin{cases} G(u), & u \leq -\varepsilon, \\ G(-\varepsilon), & -\varepsilon \leq u \leq \varepsilon, \\ G(u) - \Delta_\varepsilon, & u \geq \varepsilon. \end{cases}$$

¹¹ *Calcul des probabilités*, Paris (1925), p. 167. In the general case the integral should be considered in the sense of the Cauchy principal value.

Since the function $G_\varepsilon(u)$ is bounded and nondecreasing, we can write

$$G_\varepsilon(u) = \lambda_\varepsilon \varphi_\varepsilon(u),$$

where λ_ε is a positive number and the function $\varphi_\varepsilon(u)$ differs by an additive constant from a certain distribution (this statement becomes trivial if the total variation of $G_\varepsilon(u)$ is equal to zero for each $\varepsilon > 0$). Let further

$$\begin{aligned} f_\varepsilon(t) &= \int_{|u|>\varepsilon} (e^{itu} - 1) dG(u) = \int_{-\infty}^{+\infty} (e^{itu} - 1) dG_\varepsilon(u) = \\ &= \lambda_\varepsilon \int_{-\infty}^{+\infty} (e^{itu} - 1) d\varphi_\varepsilon(u) = \lambda_\varepsilon \{ \varphi_\varepsilon(t) - 1 \}, \end{aligned}$$

where $\varphi_\varepsilon(t)$ is the characteristic function of the distribution $\varphi_\varepsilon(x)$. Evidently, the expression

$$\frac{\lambda_\varepsilon}{n} \varphi_\varepsilon(t) + \left\{ 1 - \frac{\lambda_\varepsilon}{n} \right\}$$

is a characteristic function for each $n \geq \lambda_\varepsilon$. Hence the function

$$n \log \left\{ \frac{\lambda_\varepsilon}{n} \varphi_\varepsilon(t) + \left(1 - \frac{\lambda_\varepsilon}{n} \right) \right\} = n \log \left\{ 1 + \frac{\lambda_\varepsilon}{n} [\varphi_\varepsilon(t) - 1] \right\}$$

is the logarithm of a characteristic function. The same is true for its limit as $n \rightarrow \infty$ which is equal to

$$\lambda_\varepsilon [\varphi_\varepsilon(t) - 1] = f_\varepsilon(t).$$

Therefore, if $G(u)$ is an arbitrary bounded nondecreasing function and ε is an arbitrary positive number, then the integral

$$\int_{|u|>\varepsilon} (e^{itu} - 1) dG(u) \tag{A.6}$$

is the logarithm of a certain characteristic function. The same is valid also for the integral

$$\int_{|u|>\varepsilon} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) dG(u),$$

which differs from (A.6) only by a term $it\gamma$, where γ is a real constant. We can also change in the last integral $dG(u)$ to $(1+u^2)/u^2 dG(u)$, since the function $(1+u^2)/u^2$ is bounded for $|u| > \varepsilon$. Finally we can pass to the limit as $\varepsilon \rightarrow 0$. Therefore the function

$$\lim_{\varepsilon \rightarrow 0} \int_{|u| > \varepsilon} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) \frac{1+u^2}{u^2} dG(u)$$

is the logarithm of a characteristic function. But the expression (A.5) differs from this limit only by a term $it\gamma$ and a term of the type $-at^2$ ($a \geq 0$) which is due to a possible discontinuity of $G(u)$ at $u = 0$. Hence the function $\varphi(t)$ is the product of a characteristic function with an expression of the type

$$e^{it\gamma - at^2},$$

where γ is a real constant and $a \geq 0$. The last expression is a characteristic function of a certain Gaussian Law. Hence the function $\varphi(t)$ is a characteristic function as well.

The corresponding law is evidently infinitely divisible since $\lambda \log \varphi(t)$ is for each $\lambda \geq 0$ the expression of the same type as (A.5). Thus, by what is proved above, $\lambda \log \varphi(t)$ is the logarithm of a certain characteristic function.

Supplement. B. V. Gnedenko has pointed out that, to get the statement for the expression preceding to (A.5), one needs to see that for $\alpha \rightarrow \infty$ the limit

$$\int_{|u| \geq \alpha} dG_h(u) \rightarrow 0$$

is *uniform with respect to h* . To show this, it is sufficient to note that, analogously to (A.4), one can prove the inequality

$$-\frac{\alpha}{2} \int_0^{2/\alpha} \operatorname{Re}[I_h(t)] dt \geq \int_{|u| \geq \alpha} \left(1 - \frac{\sin(2u/\alpha)}{(2u/\alpha)} \right) dG_h(u) \geq \frac{1}{2} \int_{|u| \geq \alpha} dG_h(u).$$

The left hand-side of this inequality tends as $h \rightarrow 0$ to

$$-\frac{\alpha}{2} \int_0^{\frac{2}{\alpha}} \operatorname{Re}[\log \varphi(t)] dt,$$

which is sufficiently small for sufficiently large α .

References

- [1] Applebaum, D.: *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge (2004)
- [2] Barndorff-Nielsen, O.E. and Halgreen, C: Infinite divisibility of the hyperbolic and generalized inverse Gaussian distributions. *Z. Wahrsch. verw. Gebiete* **38** (1977) 309–311
- [3] Barndorff-Nielsen, O.E., Mikosch, T. and . Resnik, S.T. (eds): *Lévy Processes: Theory and Applications*. Birkhäuser, Boston (2001)
- [4] Baxter, G. Shapiro, J.M.: On bounded infinitely divisible random variables. *Sankhyā* **22** (1960) 253-260
- [5] Bawly, G.M.: Über einige Werallgemeinerungen der Grenzwertsätze der Wahrscheinlichkeitsrechnung. *Mat. Sbornik [Réc. Math. (Moscow) N.S.]*, **1** (43) (1936) 917-930
- [6] Bawly, G.M.: Über den lokalen Grenzwertsatz der Wahrscheinlichkeitsrechnung. *Rev. Fac. Sci. Univ. Istanbul*, New Series **2** No 2 (1937) 79-92
- [7] Bertoin, J.: *Lévy Processes*. Cambridge University Press, Cambridge (1996) [Cambridge Tracts in Mathematics, Vol. 121]
- [8] Bingham, N.H.: Fluctuation, *Mathematical Scientist* **23** (1998) 63-73.
- [9] Bingham, N.H., Goldie, C.M. and Teugels, J.L.: *Regular Variation*. Cambridge Univ. Press, Cambridge (1987)
- [10] Bingham, N.H. and Kiesel, R.: *Risk-Neutral Valuation. Pricing and Hedging of Financial Derivatives*. Springer Finance, Berlin (1998)
- [11] Blum, J.R. and Rosenblatt, M.: On the structure of infinitely divisible distributions. *Pacific J. Math.* **9** (1959) 1-7
- [12] Bondesson, L.: On the infinite divisibility of products of powers of gamma variables, *Z. Wahrsch. verw. Gebiete* **49** (1979) 171-175
- [13] Bondesson, L.: On simulation from infinitely divisible distributions, *Advances in Applied Probability* **14** (1982) 855-869

- [14] Bondesson, L.: *Generalized Gamma Convolutions and related classes of Distributions and Densities*. Lecture Notes in Statistics, Vol. 76. Springer-Verlag, Berlin (1992)
- [15] Bondesson, L., Kristiansen, G.K. and Steutel, F.W.: Infinite divisibility of random variables and their integer parts. *Statistics and Probability Letters* **28** No. 3 (1996) 271-278
- [16] Bose, A., Dasgupta, A., Rubin, H.: A contemporary review and bibliography of infinitely divisible distributions and processes. *Sankhyā Series A* **64** Pt 3 (2002) 763-819
- [17] Cantelli, F.P.: Sui confini della probabilità. *Atti del Congresso Internazionale dei Matematici*, Bologna 3-10 September 1928. Zanichelli Bologna (1932) Vol. VI (Sezione IV-A) 47-60.
- [18] Carasso, A.S.: Infinitely divisible pulses, continuous deconvolution, and the characterization of linear time invariant systems. *SIAM J. Appl. Math.* **47** (1987) 892-927
- [19] Chung, K.L.: *A Course in Probability Theory*. Academic Press, New York (1976)
- [20] Cifarelli, D.M., Regazzini, E.: de Finetti's contribution to probability and statistics. *Statistical Science* **11** (1996) 253-282
- [21] Cramer, H.: Über eine Eigenschaft der normalen Verteilungsfunktion. *Math. Zeitschr. B.* **41** (1936) 405-414
- [22] Csörgö, S., Häusler, E., Mason D.M.: A probabilistic approach to the asymptotic distribution of sums of independent identically distributed random variables. *Adv. Appl. Math.* **9** (1988) 259-333
- [23] Daboni, L.: Bruno de Finetti (Necrologio). *Boll. Unione Matematica Italiana* (Ser. VII **1-A** (1987) 283-308
- [24] Daboni, L., Sigalotti, M., Zecchin, M.: Processi stocastici: teoria ed applicazioni. In: *Atti del Convegno Ricordo di Bruno de Finetti, Professore nell'Ateneo triestino*, Trieste 30-31 maggio 1986, Università di Trieste (1987) 81-96

- [25] de Finetti, B.: *Opere Scelte* [Selected Works] Voll. I, II edited by Unione Matematica Italiana and Associazione per la Matematica Applicata alle Scienze Economiche e Sociali. Edizioni Cremonese, Roma (2006)
- [26] de Finetti, B.: Sulle funzioni ad incremento aleatorio, *Rendiconti della R. Accademia Nazionale dei Lincei* (Ser VI) **10** (1929) 163-168 [reprinted in [25] Vol. I, pp. 65-70]
- [27] de Finetti, B.: Sulla possibilità di valori eccezionali per una legge di incrementi aleatori, *Rendiconti della R. Accademia Nazionale dei Lincei* (Ser VI) **10** (1929) 325-329 [reprinted in [25] Vol. I, pp. 71-76]
- [28] de Finetti, B.: Integrazione delle funzioni ad incremento aleatorio, *Rendiconti della R. Accademia Nazionale dei Lincei* (Ser VI), **10** (1929) 548-553 [reprinted in [25] Vol. I, pp. 77-82]
- [29] de Finetti, B.: Le funzioni caratteristiche di legge istantanea, *Rendiconti della R. Accademia Nazionale dei Lincei* (Ser VI) **12** (1930) 278-282 [reprinted in [25] Vol. I, pp. 159-164]
- [30] de Finetti, B.: Funzione caratteristica di un fenomeno aleatorio. *Memorie della R. Accademia Nazionale dei Lincei*, **4** No 5 (1930) 86-133 [reprinted in [25] Vol. I, pp. 109-158]
- [31] de Finetti, B.: Le funzioni caratteristiche di legge istantanea dotate di valori eccezionali. *Rendiconti della R. Accademia Nazionale dei Lincei* (Ser VI), **14** (1931) 259-265 [reprinted in [25] Vol. I, pp. 171-178]
- [32] de Finetti, B.: Le leggi differenziali e la rinunzia al determinismo. *Rendiconti del Seminario Matematico della R. Università di Roma* **7** (1931) 63-74 [reprinted in [25] Vol. I, pp. 179-190]
- [33] de Finetti, B.: Funzione caratteristica di un fenomeno aleatorio, *Atti del Congresso Internazionale dei Matematici*, Bologna 3-10 September 1928. Zanichelli, Bologna (1932) Vol. VI (Sezione IV-A) pp. 179-190
- [34] de Finetti, B.: Funzioni aleatorie, in *Atti del I Congresso UMI*. Zanichelli, Bologna (1938), pp. 413-416 [reprinted in [25] Vol. I, pp. 401-404]

- [35] de Finetti, B.: *Theory of Probability*. 2 Voll. Wiley, New York (1975)
[English translation of the Italian edition, Einaudi, Torino (1970)]
- [36] Doob, J.L.: *Stochastic Processes*. Wiley, New York (1953)
- [37] Embrechts, P., Goldie, C. and Verarbeke, N.: Subexponentiality and infinite divisibility. *Z. Wahrsch. verw. Gebiete* **49** (1979) 335-347
- [38] Feller, W.: *An Introduction to Probability Theory and its Applications*. Vol. 2, 2-nd edn., Wiley, New York (1971) [1-st edn. 1966]
- [39] Fisz, M.: Infinitely divisible distributions: recent results and applications. *Ann. Math. Statist.* **33** (1962) 68-84
- [40] Gawronski, R., Rachev, S.T.: Stable models in testable asset pricing. In: *Approximation, Probability, and Related Fields* (Santa Barbara, 1993), G. Anastassiou and S.T. Rachev eds., Plenum Press, New York (1994)
- [41] Gerber, H.U.: On the probability of ruin for infinitely divisible claim amount distributions. *Insurance Math. Econom.* **11** (1992) 163-166
- [42] Geluk, J.L., de Haan, L.: Stable probability distributions and their domains of attraction: a direct approach. *Probab. Math. Statist.* **30** (2000) 169-188
- [43] Gnedenko, B.V.: On a characteristic property of infinitely divisible distributions. *Bulletin of the Moscow State University* **5** (1937) 10-16 [in Russian]
- [44] Gnedenko, B.V.: Alexander Iacovlevich Khinchin, In: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, held at the Statistical Laboratory, University of California at Berkeley, June 20 - July 30, 1960, Vol. II, University of California Press, Berkeley and Los Angeles (1961), pp. 10-15.
- [45] Gnedenko, B.V. and Kolmogorov, A.N.: *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Cambridge (Mass) (1954) [English translation from the Russian edition, G.I.T.T.L., Moscow (1949)]
- [46] Grosswald, D.N.: The Student t-distribution of any degree of freedom is infinite divisible. *Z. Wahrsch. verw. Gebiete* **36** (1976) 103-109

- [47] Goldie, C.: A class of infinitely divisible random variables. *Math. Proc. Cambridge Phil. Soc.* **63** (1967) 1141-1143
- [48] Hansen, B.G. *Monotonicity Properties of Infinitely Divisible Distributions*. CWI (Centrum voor Wiskunde en Informatica) Tracts No 69, Amsterdam (1990)
- [49] Hoffman-Jørgensen, J.: *Probability with a View Towards Statistics*, Vol. I, Chapman & Hall, New York - London (1994) [Chapman & Hall Probability Series]
- [50] Ibragimov, I. and . Linnik, Y.: *Independent and Stationary Sequences of Random Variables*. Addison-Wesley, Reading, Massachusetts (1968)
- [51] Ismail, M.E.H.: Bessel functions and the infinitely divisibility of the Student t -distribution, *Ann. Probab.* **5** (1977) 582-585.
- [52] Ismail, M.E.H. and Kelker, D.H.: Special functions, Stieltjes transforms and infinite divisibility, *SIAM J. Math. Anal.* **10** No 5 (1979) 884-901
- [53] Ismail, M.E.H. and May C.P.: Special functions, infinite divisibility and transcendental equations. *Math. Proc. Cambridge Phil. Soc.* **85** (1979) 453-464
- [54] Ismail, M.E.H. and . Miller, K.S.: An infinitely divisible distributions involving modified Bessel functions. *Proc. Amer. Math. Soc.* **85** (1982) 233-238
- [55] Itô, K.: On stochastic processes I (Infinitely divisible laws of probability). *Jap. J. Math.* **18** (1942) 261-301
- [56] Itô, K. and McKean H.P.: *Diffusion Processes and Their Sample Paths*. Springer, Berlin (1996)
- [57] Jona-Lasinio, G.: The renormalization group: a probabilistic approach. *Il Nuovo Cimento B* **26** (1975) 99-119
- [58] Kahane, J.-P.: Definition of stable laws, infinitely divisible laws, and Lévy processes, in: M.F. Shlesinger, G.M. Zaslavsky and U. Frisch (Editors), *Lévy Flights and Related Topics in Physics*. Springer Verlag, Berlin (1995), pp. 99-109 [Lecture Notes in Physics, No 450]

- [59] Khintchine, A.Ya.: Sulle successioni stazionarie di eventi. *Giornale dell'Istituto Italiano degli Attuari* **3** (1932) 267-272
- [60] Khintchine, A.Ya.: Sul dominio di attrazione della legge di Gauss. *Giornale dell'Istituto Italiano degli Attuari* **6** (1935) 378-393
- [61] Khintchine, A.Ya.: Su una legge dei grandi numeri generalizzata. *Giornale dell'Istituto Italiano degli Attuari* **7** (1936) 365-377
- [62] Khintchine, A.Ya.: A new derivation of a formula by P. Lévy. *Bulletin of the Moscow State University* **1** (1937) 1-5 [in Russian]
- [63] Khintchine, A.Ya.: On arithmetic of distributions. *Bulletin of the Moscow State University* **1** (1937) 6-17 [in Russian]
- [64] Khintchine, A.Ya.: Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze, *Mat. Sbornik [Réc. Math. (Moscow) N.S.]* **2** (44) (1937) 79-119
- [65] Khintchine, A.Ya.: *Limit Distributions for the Sum of Independent Random Variables*. O.N.T.I., Moscow (1938), pp. 115. [in Russian]
- [66] Khintchine, A. and Lévy, P.: Sur les lois stables, *C.R. Acad. Sci. Paris* **202** (1936) 374-376
- [67] Kolmogorov, A.N.: Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, *Math. Ann.* **104** (1931) 415-458
- [68] Kolmogorov, A.N.: Sulla forma generale di un processo stocastico omogeneo (Un problema di Bruno de Finetti). *Rendiconti della R. Accademia Nazionale dei Lincei* (Ser. VI), **15** (1932) 805-808
- [69] Kolmogorov, A.N.: Ancora sulla forma generale di un processo stocastico omogeneo. *Rendiconti della R. Accademia Nazionale dei Lincei* (Ser. VI), **15** (1932) 866-869
- [70] Kolmogorov, A.N.: On the general form of a homogeneous stochastic process (The problem of Bruno de Finetti), paper No 13. In: A.N. Shiryaev (ed). *Selected Works of A.N. Kolmogorov, Vol 2: Probability Theory and Mathematical Statistics*. Kluwer, Dordrecht (1992), pp. 121-127. The comment by Zolotarev is at pp. 528-530

- [71] Lévy, P.: *Calcul des Probabilités*, Gauthier-Villars, Paris (1925), pp. viii+350. Reprinted in 2003 by Editions Jaques Gabay (Les Grands Classiques Gauthier-Villars)
- [72] Lévy, P.: Fonctions à croissance régulière et iteration d'ordre fractionnaire, *Ann. Math. Bologna*, (Ser. 4) **6** (1928-29) 269-298
[reprinted as [58] in *Lévy's OEUVRES*, pp. 219-248. It contains the communication at the Int. Congress of Mathematicians, Bologna 1928 and two notes published in CRAS in 1926 and 1927]
- [73] Lévy, P.: Sur les intégrales dont les éléments sont des variables aléatoires indépendentes. *Ann. R. Scuola Norm. Pisa* (Ser. II) **3** (1934) 337-366
- [74] Lévy, P.: Observation sur un précédent mémoire de l'auteur. *Ann. R. Scuola Norm. Pisa* (Ser. II) **4** (1935) 217-218
- [75] Lévy, P.: *Théorie de l'Addition des Variables Aléatoires*, 2nd edn. Gauthier-Villars, Paris (1954) [1st edn, 1937] reprinted in 2003 by Editions Jaques Gabay, Paris (Les Grands Classiques Gauthier-Villars)
- [76] Lévy, P.: *Quelques aspects de la pensée d'un mathématicien*. Albert Blanchard, Paris (1970)
- [77] Lévy, P.: *Oeuvres de Paul Lévy*. Edited by D. Dugué, P. Deheuvels & M. Ibéro, 6 Volumes, Gauthier-Villars, Paris (1973-1980)
- [78] Loève, M.: Paul Lévy, 1886-1971, *Ann. Probab.* **1** (1973) 1-18
- [79] Loève, M.: *Probability Theory*, Vol. I, II, 4th ed. Springer Verlag, Berlin, (1977-1978)
- [80] Lukacs, E.: *Characteristic Functions*. 2-nd edition, Griffin, London (1970)
- [81] Lukacs, E.: *Developments in Characteristic Function Theory*. Macmillan, New York (1983)
- [82] Lundberg, F.: Thesis on *Approximerad Framställning av Sannolikehetsfunktioner. Aterförsäkering av Kollektivrisker*. Almqvist and Wiksell, Uppsala (1903)

- [83] Mandelbrot, B.B.: The variation of certain speculative prices. *J. Business* **36** (1963) 394-419.
- [84] Marcus, M.B. and Pisier, G.: Some results on continuity of stable processes and the domain of attraction of continuous stable processes, *Ann. Inst. H. Poincaré Prob. Statist.* **20** (1984) 177-199
- [85] *Mathematics in Moscow University*. Collection of scientific articles (K.A. Rybnikov ed.). Moscow State University (1992)
- [86] *Mathematics in USSR for thirty years. 1917-1947* (A.G. Kurosh, A.I. Markushevich, P.K. Rashevskij eds.). Moscow-Leningrad: GosTechTheoretIzdat (1948)
- [87] Mittnik, S. and Rachev, S.T.: Stable distributions for asset returns. *Appl. Mathematics Letters* **2** No. 3 (1989) 301-304
- [88] Mittnik, S. and Rachev, S.T.: Modeling asset returns with alternative stable models. *Econometric Rev.* **12** (1993) 261-330.
- [89] Mittnik, S. and Rachev, S.T.: Reply to comments on “Modeling asset returns with alternative stable models” and some extensions. *Econometric Rev.* **12** (1993) 347-389
- [90] Mittnik, S. and Rachev, S.T.: Stable Modelling in Finance. *Mathematical and Computer Modelling*, Special Issue (1997)
- [91] Novikov, E.A.: Infinite divisible distributions in turbulence. *Phys. Rev. E* **50** (1994) R3303-R3305
- [92] Paley, R.E., Wiener, N. and Zygmund, A.: Note on random functions. *Math. Z.* **37** (1933) 647-668
- [93] Petrov, V.V.: *Sums of Independent Random Variables*. Springer, Berlin (1975) [Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 82]
- [94] Petrov, V.V.: *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*. Oxford University Press, Oxford New York (1995) [Oxford Studies in Probability, Vol. 4]
- [95] Rachev, S.T.: *Probability Metrics and the Stability of Stochastic Models*. Wiley, Chichester (1991)

- [96] Rachev, S.T and Mittnik, S.: *Stable Paretian Models in Finance*. Wiley, Chichester (2000)
- [97] Raikov, D.: On the decomposition of Poisson laws. *Comptes Rendus de l'Acad. Sci. de l'URSS* **XIV** (1937) 9-11
- [98] Rényi, A.: *Probability Theory*. North-Holland, Amsterdam (1970) [North-Holland Series in Applied Mathematics, Vol. 10]
- [99] Resnik, S.I. *Extreme Values, Regular Variations and Point Processes*. Springer, Berlin (1987)
- [100] Rosiński, J.: On series representation of infinitely divisible random vectors. *Ann. Probab.* **18** (1990) 405-430
- [101] Rosiński, J.: On the structure of stationary stable processes. *Ann. Probab.* **23** (1995) 1163-1187
- [102] Samorodnitsky, G. and Taqqu, M.: Stochastic monotonicity and Slepian-type inequalities for infinitely divisible and stable random vectors. *Ann. Probab.* **21** (1993) 143-160
- [103] Samorodnitsky, G. and Taqqu, M.: *Stable non-Gaussian Random Processes*. Chapman & Hall, New York (1994)
- [104] Satheesh, S.: A supplement to the Bose-Dasgupta-Rubin (2002) review of infinitely divisible laws and processes.
E-print (2003), pp. 20 <http://arXiv.org/abs/math.PR/0305126>
- [105] Sato, K.: *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge (1999) [Cambridge Studies in Advanced Mathematics, Vol. 68]
- [106] Shanbhagand, D.N. and Sreehari, M.: On certain self-decomposable distributions. *Z. Wahrsch. verw. Gebiete* **38** (1977) 217-222
- [107] Shanbhagand, D.N. and Sreehari, M.: An extension of Goldie's result and further results in infinite divisibility. *Z. Wahrsch. verw. Gebiete* **47** (1979) 19-25
- [108] Shiryaev, A.N. (ed): *Selected Works of A.N. Kolmogorov. Vol 2: Probability Theory and Mathematical Statistics* Kluwer, Dordrecht (1992)

- [109] Shiryaev, A.N.: Andrei Nikolaevich Kolmogorov (April 25, 1903 to October 20, 1987). A biographical sketch of his life and creative paths. In: Kolmogorov in Perspective, History of Mathematics, Vol 20, AMS (Amer. Math. Soc.) and LMS (London Math. Soc.), Rhode Island (2000), pp. 1-89. [Translated from Russian]
- [110] Steutel, F.W.: Some recent results in infinite divisibility, *Stochastic Process. Appl.* **1** (1973) 125-143
- [111] Steutel, F.W.: Infinite divisibility in theory and practice. *Scand. J. Statist.* **6** (1979) 57-64
- [112] Steutel, F.W., van Harn, H.: *Infinite Divisibility of Probability Distributions on the Real Line*. Marcel Dekker Inc., New York Basel (2004) [Series on Pure and Applied Mathematics, Vol. 259]
- [113] Surgailis, D.: On infinite divisible self-similar random fields, *Z. Wahrsch. verw. Gebiete* **58** (1981) 453-477
- [114] Takano, K.: On infinite divisibility of normed product of Cauchy densities, *J. Comput. Appl. Math.* **150** (2003) 253-263
- [115] Tucker, H.G.: *A Graduate Course in Probability*. Academic Press, New York London (1967) [Series on Probability and Mathematical Statistics, Vol. 2]
- [116] Uchaikin, V.V. and Zolotarev, V.M.: *Chance and Stability: Stable Laws and their Application*. VSP, Utrecht (1999) [Series on Modern Probability and Statistics, Vol. 3]
- [117] Van Harn, H.: *Classifying infinitely divisible distributions by functional equations*. Mathematische Centrum, Amsterdam (1978) [Mathematical Centre Tracts, Vol. 103]
- [118] Wolfe, S.J.: On the unimodality of L functions. *Ann. Math. Statist.* **42** No 3 (1971) 912-918
- [119] Wolfe, S.J.: On moments of infinitely divisible distributions, *Ann. Math. Statist.* **42** (1971) 2036-2043
- [120] Wolfe, S.J.: On the continuity properties of L functions. *Ann. Math. Statist.* **42** (1971) 2064-2073

- [121] Yamazato, M.: Unimodality of infinitely divisible distributions of class *L*. *Ann. Probability* **10** (1978) 523-531
- [122] Zolotarev, V.M.: Distribution of the superposition of infinitely divisible processes. *Theor. Prob. Appl.* **3** (1958) 185-188
- [123] Zolotarev, V.M.: Asymptotic behavior of distributions of processes with independent increments, *Theor. Prob. Appl.* **10** (1965) 28-44.
- [124] Zolotarev, V.M.: *One-dimensional stable distributions*. Amer. Math. Soc., Providence, R.I. (1986) [English Transl. from the Russian edition: *Odnomernye Ustoichivye Raspredeľniia*, Nauka, Moscow (1982)]
- [125] Zolotarev, V.M.: *Modern Theory of Summation of Random Variables*. VSP, Utrecht (1997) [Series “Modern Probability and Statistics”, No 1]